Singular Optimal Atmospheric Rocket Trajectories

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Presently studied is the problem of the ascent and acceleration of a vehicle in atmospheric flight in which a variable-thrust arc forms a part of the optimal trajectory. A two-parameter family of singular arcs was generated for time-range-fuel problems of an ascending rocket, using the modeling of Zlatskiy and Kiforenko. The short-term optimality of singular subarcs has been checked in terms of certain necessary conditions: the classical Clebsch condition, the Kelley condition or the Generalized Legendre-Clebsch condition, and the Goh condition. All of these are found to be satisfied computationally for all the candidates. The calculations were repeated for simplified thrust-along-the path modeling and similar results on optimality were obtained.

Introduction

A SINGULAR extremizing arc for the Mayer-Bolza Problem is defined as one for which the classical Legendre-Clebsch necessary condition is satisfied marginally, i.e., not with strict inequality.[1,2] In particular, if the system Hamiltonian is linear and stationary in a control variable, then the extremal is singular.

Thrust-programming for the vertical flight of a rocket was the earliest trajectory optimization studied.[3-5] It consisted of maximizing the altitude for a given propellant allocation with open final time. A variable-thrust singular subarc forms a segment of the optimal trajectory. For the case of low-altitude launch from rest, bounded thrust, and drag square law with velocity, the solution consists of an initial maximum-thrust burn joined to a variable-thrust burn followed by a final coast.[4,5] With the drag coefficient being Mach-dependent, the solution is more complex and may feature an additional maximum-thrust arc during acceleration through the transonic drag rise.[6] Problems involving singular control attracted particular attention in connection with maneuvers of a rocket in vacuum flight, the Lawden problem.[7] Singular subarcs appearing as part of an optimal trajectory appear also in cases of horizontal atmospheric flight[8,9] and ballistic atmospheric flight.[10]

In the present research, a specific problem is examined, namely, the ascent and acceleration of a rocket vehicle in atmospheric flight. The magnitude of thrust, direction of thrust, and angle of attack of the vehicle are the control variables. The magnitude of thrust appears linearly in the system Hamiltonian, thus giving rise ot singular-subarc possibilities. The notation, reference solution, and numerical data are all taken from Ref. 11; this publication inspired the present work and provided the point of departure.

The numerical solution of singular optimal control problems poses considerable difficulty by either direct or indirect methods. The difficulties include the prediction of the sequence of arcs and the times of entry and exit from the singular subarcs when the indirect method is used. An effective approach has been developed in Ref. 11 to solve the boundary-value problem. This is recapitulated herein on account of the inaccessibility of Ref. 11. The aerodynamic data of the vehicle, variation of atmospheric properties with altitude, and values for the control bound were taken from Ref. 12.

Once the boundary-value problem is solved, with the control variables satisfying the Pontryagin Principle, certain necessary conditions such as the Kelley condition or the Generalized Legendre-Clebsch condition are to be checked along the singular subarc to confirm the optimality of the extremal over short lengths of arc. When one or more nonsingular controls appear in the system dynamics, other necessary conditions obtained from the theory of the second variation for the class of singular problem, termed collectively the Goh condition, must be satisfied along the singular subarc for optimality.

The strong form of the classical Clebsch condition is not satisfied for singular arcs. The classical theorems giving sufficient conditions for optimality require the strong form of the Clebsch condition and are therefore inapplicable to composite solutions including singular subarcs. In particular, the Jacobi necessary condition that the extremal must not include a pair of conjugate points does not apply to candidates with singular arcs.[13] The present work examines state-Euler solutions for the vehicle trajectory model of Ref. 11 which satisfy the initial, final, and transversality conditions and checks the classical Clebsch condition, the Kelley condition or the Generalized Legendre-Clebsch condition, and the Goh condition along the singular subarc. Families of singular arcs have also been generated for related problems featuring constraints on final time and range, and these necessary conditions were checked to ensure the optimality of representative members of each family over short lengths of arc.

Problem Formulation

The problem is to find the trajectory of a vehicle from an assigned initial state to a final state with maximum horizontal velocity at the terminal point of the trajectory. The magnitude of thrust, the thrust-vector angle, and the angle of attack are the three control variables, each with lower and upper bounds. The following assumptions are made: point-mass model, Newtonian central gravitational field, two-dimensional trajectory, air density varies exponentially with altitude, drag coefficient approximated by a quadratic polar curve, lift varies linearly with angle of attack, and constant exhaust velocity.

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The nondimensional equations of motion for the system are as follows:

$$\dot{r}=v\sin\theta$$

$$\dot{\varphi}=\frac{v\cos\theta}{r}$$

$$\dot{v}=\frac{(P\cos\gamma-D)}{m}-r^{-2}\sin\theta$$

$$\dot{\theta}=\frac{(P\sin\gamma+L)}{mv}+(r^{-1}v-r^{-2}v^{-1})\cos\theta$$

$$\dot{m}=-\frac{P}{c} \tag{1}$$

where r is the radial coordinate, φ the range angle, v the velocity of the vehicle, θ the flight path angle, m the mass of the vehicle, P the thrust, D the aerodynamic drag, L the aerodynamic lift, c the exhaust velocity, α the angle of attack, and γ the thrust-vector angle. Nondimensionalization was performed by the following factors:

$$\hat{r}=R_e,\quad \hat{t}=G^{-1/2}\hat{r}^{3/2},\quad \hat{v}=G^{1/2}\hat{r}^{-1/2}$$

$$\hat{g}=G\hat{r}^{-2},\quad \hat{m}=m_0$$

Here, R_e denotes the radius of the Earth, G the gravitational constant, $\hat{g}$ the acceleration due to gravity at the surface of the Earth, and $\hat{m}$ the launching mass of the vehicle. The thrust, lift, and drag are nondimensionalized by $\hat{m}\hat{g}$.

The variation of drag and lift with altitude, velocity, and angle of attack are given by the following set of equations:

$$D=C_D bv^2\exp[\beta(1-r)] \tag{2}$$

where $C_D=C_{D0}+B\alpha^2$

$$L=C_L bv^2\exp[\beta(1-r)]$$

where $C_L=A\alpha$. A and B are coefficients of a quadratic function approximating a polar curve. The angle of attack is denoted by α, and b and β are constants. The drag coefficient, lift coefficient, and zero-lift drag coefficient are denoted by C_D, C_L, and C_{D0}, respectively.

The initial conditions for the five state variables were specified as r_0, φ_0, v_0, θ_0, and m_0. The final conditions for the state variables were specified as r_f, θ_f, and m_f. The problem is to maximize the velocity v_f at termination of the trajectory for the specified final conditions. Since the range angle is neither specified nor to be optimized, φ_f is free at terminal time. The values of state at initial and terminal times have been taken from Ref. 11. The lower and upper bounds on the control variables are given by the following set of inequality constraints:

$$0\le P\le P_{\max},\quad |\alpha|\le\alpha_0,\quad |\gamma|\le\gamma_0 \tag{3}$$

where $P_{\max}$, α_0, *and* γ_0 are assigned constants determining the control limits.

The state variables are piecewise-differentiable functions of time, while the control functions may exhibit jump discontinuities. The modeling featuring thrust direction controllable separately from the angle of attack is more than a little unusual for a rocket vehicle. It and the notation, unusual in the West, are from Refs. 11 and 12. A simplified model with thrust along the path will also be examined in the sequel.

Variational Hamiltonian

An approach to solving this problem is the formation of a variational Hamiltonian and application of the Pontryagin Principle.[14] The Hamiltonian is

$$\mathrm{H}\equiv\Sigma\lambda_i\dot{x}_i \tag{4}$$

where λ_i, $i=1,2,\ldots5$ are the costate or adjoint variables and x_i, $i=1,2,\ldots5$ are the state variables.

The Hamiltonian H can be represented as

$$\mathrm{H}=H_0+PH_1 \tag{5}$$

where H_1 is the switching function. H_0 and H_1 take the following form:

$$H_0=\lambda_r v\sin\theta+\lambda_\varphi r^{-1}v\cos\theta-\lambda_v(m^{-1}D+r^{-2}\sin\theta)$$
$$+\lambda_\theta v^{-1}m^{-1}L+\lambda_\theta\cos\theta(r^{-1}v-r^{-2}v^{-1}) \tag{6a}$$

$$H_1=\lambda_v m^{-1}\cos\gamma+\lambda_\theta v^{-1}m^{-1}\sin\gamma-\lambda_m c \tag{6b}$$

The Euler-Lagrange equations define the differential system of costate variables:

$$\dot{\lambda}_i=-\frac{\partial\mathrm{H}}{\partial x_i} \tag{7}$$

The components of costate vector λ satisfying the preceding equations are

$$\dot{\lambda}_r=\lambda_\varphi vr^{-2}\cos\theta+\lambda_v\left(m^{-1}\frac{\partial D}{\partial r}-2r^{-3}\sin\theta\right)$$
$$+\lambda_\theta\left[(vr^{-2}-2v^{-1}r^{-3})\cos\theta-v^{-1}m^{-1}\frac{\partial L}{\partial r}\right]$$

$$\dot{\lambda}_\varphi=0$$

$$\dot{\lambda}_v=G_1+\lambda_\theta m^{-1}v^{-2}P\sin\gamma$$

$$\dot{\lambda}_\theta=G_2-\lambda_\theta v^{-1}(m^{-1}D+r^{-2}\sin\theta)$$

$$\dot{\lambda}_m=m^{-1}\left[P\left(H_1+\frac{\lambda_m}{c}\right)+G_3\right]$$

where G_1, G_2, and G_3 are formed for convenience in evaluating the singular control. These are defined as follows:

$$G_1=-\lambda_r\sin\theta-\lambda_\varphi r^{-1}\cos\theta+\lambda_v m^{-1}\frac{\partial D}{\partial v}$$
$$+\lambda_\theta\left[m^{-1}v^{-2}\left(L-v\frac{\partial L}{\partial v}\right)-r^{-1}\cos\theta(1+r^{-1}v^{-2})\right]$$

$$G_2=-\lambda_r v\cos\theta+\lambda_\varphi vr^{-1}\sin\theta+\lambda_v r^{-2}\cos\theta$$
$$+\lambda_\theta(m^{-1}v^{-1}D+vr^{-1}\sin\theta)$$

$$G_3=m^{-1}(v^{-1}L\lambda_\theta-D\lambda_v)$$

Equations (1) and (7) together constitute the state-Euler system.

Control Logic

The optimal values of the control variables are generally to be determined from the Pontryagin Principle. Since the control P appears linearly in the Hamiltonian, these do not determine optimal thrust. Since P is bounded, the following logic provides the maximum of the Hamiltonian:

$$P = P_{\max} \quad \text{when} \quad H_1 > 0$$

$$P = P_0 \quad \text{when} \quad H_1 = 0$$

$$P = 0 \quad \text{when} \quad H_1 < 0$$

where P_0 is the singular control for the magnitude of thrust. The bound on singular control is given by

$$0 \le P_0 \le P_{\max}$$

The optimal values of the other two control variables, thrust-vector angle γ and angle of attack α, are obtained by maximizing the Hamiltonian, i.e.,

$$\frac{\partial H}{\partial \gamma} = 0 \quad \text{and} \quad \frac{\partial H}{\partial \alpha} = 0$$

When the magnitudes of these control variables, which give a stationary value of the Hamiltonian, go beyond the assigned limits, the respective bound values are chosen so as to maximize the Hamiltonian. The logic is as follows:

$$\gamma = \hat{\gamma} \quad \text{when} \quad |\hat{\gamma}| < \gamma_0$$

$$\gamma = \gamma_0 \quad \text{when} \quad |\hat{\gamma}| \ge \gamma_0 \text{ and } \lambda_\theta > 0$$

$$\gamma = -\gamma_0 \quad \text{when} \quad |\hat{\gamma}| \ge \gamma_0 \text{ and } \lambda_\theta < 0$$

and

$$\alpha = \hat{\alpha} \quad \text{when} \quad |\hat{\alpha}| < \alpha_0 \text{ and } \lambda_v > 0$$

$$\alpha = \alpha_0 \quad \text{when} \quad (|\hat{\alpha}| \ge \alpha_0 \text{ or } \lambda_v < 0) \text{ and } \lambda_\theta > 0$$

$$\alpha = -\alpha_0 \quad \text{when} \quad (|\hat{\alpha}| \ge \alpha_0 \text{ or } \lambda_v < 0) \text{ and } \lambda_\theta < 0$$

where $\hat{\gamma}$ and $\hat{\alpha}$ were derived from the maximum principle.

$$\hat{\gamma} = \arctan\left(\frac{\lambda_\theta}{v\lambda_v}\right) \tag{8}$$

$$\hat{\alpha} = \frac{A}{2B}\frac{\lambda_\theta}{v\lambda_v} \tag{9}$$

Transversality Conditions

The performance index to be maximized is the terminal velocity. The range angle φ_f is not specified. Therefore, the tansversality conditions give the following terminal values for the costate variables λ_v and λ_φ:

$$\lambda_v(t_f) = 1 \tag{10}$$

$$\lambda_\varphi(t_f) = 0 \tag{11}$$

Since the Hamiltonian is not an explicit function of time, H = a constant. Furthermore, for final time unspecified, we have the transversality condition

$$H(t_f) = 0 \tag{12}$$

Thus, we have $H = 0$ along the optimal trajectory.

Evaluation of Singular Control

When the switching function H_1 becomes zero in the interval $(\tau,\nu) \subset (t_0,t_f)$, the control corresponding to the magnitude of thrust is singular. In these circumstances, there are finite control variations of P which do not affect the value of the Hamiltonian. It follows that the Pontryagin Principle does not directly determine a unique optimal control as a function of state and costate variables.

The principle requires $\partial H/\partial P$ to vanish identically and a sequence of conditions to follow, namely

$$\dot{H}_P = 0, \quad \ddot{H}_P = 0 \text{ etc.} \tag{13}$$

The original control $P(t)$ is determined from the equation

$$\frac{d^{2k}}{dt^{2k}}\left(\frac{\partial H}{\partial P}\right) = 0 \tag{14}$$

where k is the smallest integer for which P enters explicitly into the left-hand side of the preceding equation. The order of the singular extremal is denoted by k.

In this specific problem, the first equation of Eqs. (13) takes the following form:

$$\dot{H}_1 = m^{-1}[G_1 \cos\gamma + G_2 v^{-1} \sin\gamma - (G_3/c)] \tag{15}$$

The value of the singular control P_0 is determined from the equation

$$\frac{d^2 H_1}{dt^2} = KP_0 + Q = 0 \tag{16}$$

and, therefore, for the first-order singular extremal we obtain the singular control

$$P_0 = -Q/K \tag{17}$$

The values of Q and K were derived by differentiating the switching function twice with respect to time and substituting the state and Euler-Lagrange equations to obtain the singular control in terms of the state variables, costate variables, and nonsingular control variables.

Necessary Conditions for Optimality

The classical Clebsch condition, the Generalized Legendre-Clebsch condition or Kelley condition, the Goh condition, and the Robbins' equality condition are to be checked for the candidate extremal.

Classical Clebsch Condition

A singular extremal has the property that, at each point of the arc, there is some admissible weak control variation

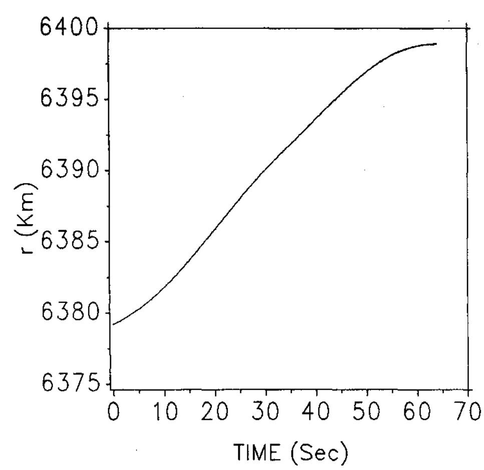

Fig. 1 Variation of radius with time.

that leaves the value of the variational Hamiltonian unchanged to second order. The matrix H_{uu} (whose elements are second partial derivatives of the variational Hamiltonian with respect to components of control vector u) is singular along the arc. In a form compatible with the Pontryagin Principle, it is required that the classical Clebsch condition hold in a weak sense, i.e., that the matrix H_{uu} be negative semidefinite.

$$H_{uu}=\begin{bmatrix} H_{\alpha\alpha} & H_{\alpha\gamma} & H_{\alpha P} \\ H_{\gamma\alpha} & H_{\gamma\gamma} & H_{\gamma P} \\ H_{P\alpha} & H_{P\gamma} & H_{PP} \end{bmatrix} \leq 0 \tag{18}$$

For the specific problem at hand, the following apply along any extremal:

$$H_{\alpha\gamma}=H_{\gamma\alpha}=0, \quad H_{\alpha P}=H_{P\alpha}=0$$

$$H_{\gamma P}=H_{P\gamma}=0, \quad H_{PP}=0$$

$$H_{\alpha\alpha}=-2Bbm^{-1}v^2\exp[\beta(1-r)]\lambda_v$$

$$H_{\gamma\gamma}=-Pm^{-1}(\lambda_v \cos\gamma+v^{-1}\lambda_\theta \sin\gamma)$$

This yields a simplified diagonal matrix that must be negative semidefinite along the extremal for optimality. Hence, the diagonal elements must be negative or zero. This produces two inequality conditions:

$$\lambda_v \geq 0 \tag{19}$$

$$\lambda_v \cos\gamma+\lambda_\theta v^{-1} \sin\gamma \geq 0 \tag{20}$$

Kelley Condition

Kelley (1964) deduced a new necessary condition for singular optimal control by studying the second variation under a special control variation.[15] Kelley's result was generalized by Tait (1965), Kopp and Moyer (1965), and Robbins (1967) to give the generalized Legendre-Clebsch condition.[13,15,16]

The condition may be stated as follows:

$$(-1)^k \frac{\partial}{\partial P}\left[\frac{d^{2k}\partial H}{dt^{2k}\partial P}\right] \leq 0 \tag{21}$$

where k is the order of singularity of the arc. The smallest positive integer for which the control P appears explicitly in the left-hand side of Eq. (16) is k. Equation (21) reduces to

$$K \geq 0 \tag{22}$$

For the present problem, the order of the singular arc is one.

Goh Condition

This necessary condition applies to systems with multiple controls when singular behavior appears. It is required that a certain matrix be symmetric, and if so, it is required that another matrix be negative semidefinite.[17] The negative semidefiniteness of the diagonal terms of the matrix imposes the same conditions as those obtained for the classical Clebsch condition and Kelley condition.

The following matrices are defined:

$$R_1=\begin{bmatrix} H_{\alpha\alpha} & 0 \\ 0 & H_{\gamma\gamma} \end{bmatrix}$$

If the state equations are represented in the form $\dot{x}_i=f_i(x_i,u,t)$, $i=1,2,\ldots 5$, we define the matrices

$$B_1=\left[\frac{\partial f_i}{\partial\alpha} \;\; \frac{\partial f_i}{\partial\gamma}\right]$$

where $\partial f_i/\partial\alpha$ and $\partial f_i/\partial\gamma$ are column vectors.

$$B_2=\left[\frac{\partial f}{\partial P}\right]$$

where $\partial f/\partial P$ is a column vector. Matrices Q_1 and Q_2 are defined as

$$Q_1=\begin{bmatrix} H_{\alpha r} & H_{\alpha\varphi} & H_{\alpha v} & H_{\alpha\theta} & H_{\alpha m} \\ H_{\gamma r} & H_{\gamma\varphi} & H_{\gamma v} & H_{\gamma\theta} & H_{\gamma m} \end{bmatrix}$$

$$Q_2=[H_{Pr} \;\; H_{P\varphi} \;\; H_{Pv} \;\; H_{p\theta} \;\; H_{Pm}]$$

and a 5×5 matrix P_1 is given by

$$P_1=[H_{x_ix_i}]$$

We define a 5×5 matrix A given by

$$A=\left[\frac{\partial f_i}{\partial x_i}\right]$$

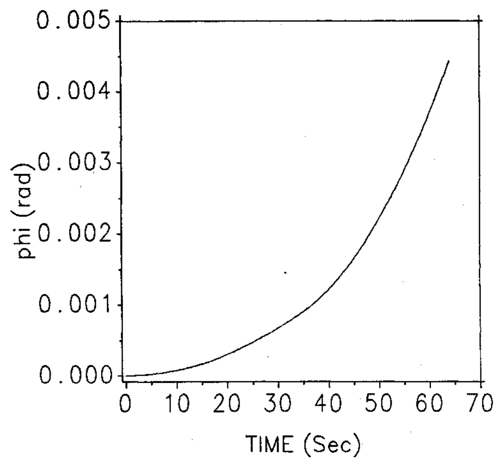

Fig. 2 Variation of range angle with time.

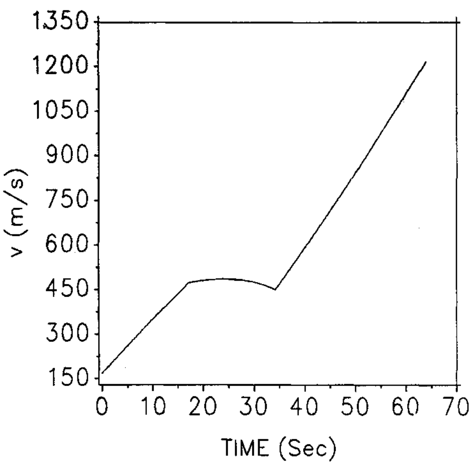

Fig. 3 Variation of velocity with time.

Thereafter we define the matrices R_2 and R_3

$$R_2 = B_2^T Q_1^T - Q_2 B_1$$

where R_2 is a 1×2 matrix and

$$R_3 = B_2^T P_1 B_2 - \frac{\mathrm{d}}{\mathrm{d}t} Q_2 B_2 - Q_2 B_3 - B_3^T Q_2^T$$

where R_3 is a 1×1 matrix and $B_3 = AB_2 - \dot{B}_2$.

By the Goh condition, we first check the symmetry of matrix $Q_2 B_2$, which is trivially satisfied since it is a 1×1 matrix. The augmented matrix R is formed from matrices R_1, R_2, and R_3

$$R = \begin{bmatrix} R_1 & R_2^T \\ R_2 & R_3 \end{bmatrix}$$

where R is a 3×3 matrix.

The Goh condition requires that the matrix R be negative semidefinite. This can be ensured by checking the signs of eigenvalues of the matrix. The matrix R_1 must be negative semidefinite, which coincides with the classical Clebsch condition, and the negative semi-definiteness of R_3 is equivalent to the Kelley condition along the singular arc.

Robbins' Equality Condition

A general expression for Robbins' equality condition[13] valid for singular cases of linear-control type with the nonsingular controls eliminated by the Pontryagin Principle can be stated as follows:

$$H_{u\lambda} H_{xu} - H_{ux} H_{\lambda u} = 0 \quad \text{for all } \lambda \text{ and } x$$

where u denotes the controls appearing linearly in the variational Hamiltonian. In the present problem, there exits only a single element P in the manifold u; hence, the condition is trivially satisfied.

Numerical Solution and Results

Numerical Data

The aerodynamic data and the control bounds for the vehicle powered by a rocket engine have been adopted from Ref. 12. The values are $C_{D_0} = 0.05$, $A = 0.70$, $B = 0.70$, $c = 0.50$, $\beta = 500$, $P_{max} = 3.0$, $\alpha_0 = 0.2618$, and $\gamma_0 = 0.5236$. These correspond roughly to the Soviet SA-2 surface-to-air missile, NATO code-name GUIDELINE.[19] Launch is nearly vertical. The nondimensionalized initial and final values of the state variables are as follows:[11]

$r(t_0) = 1.00016$ (6379.20 km)	and $r(t_f) = 1.00325$ (6398.90 km)
$\varphi(t_0) = 0.0$	and $\phi(t_f) = \varphi_{\text{optimal}}$
$v(t_0) = 0.021$ (171.07 m/s)	and $v(t_f) = v_{max}$
$\theta(t_0) = 1.48$	and $\theta(t_f) = 0.0$
$m(t_0) = 1.0$	and $m(t_f) = 0.60$

Selection of Initial Costate Variables

The numerical solution of the boundary-value problem is performed by the selection of missing costate values and varying switching times. At the initial time t_0, we have $\lambda_\varphi(t_0) = 0$, since from transversality conditions $\lambda_\varphi(t_f) = 0$ and from Euler-Lagrange equations $\dot{\lambda}_\varphi = 0$. Therefore, there are four components of the costate vector to be determined at initial time t_0.

Since the costate system is linear and homogeneous, one of the components of the costate vector controls scaling. Moreover, one has $H(t) = 0$ along the trajectory, making it possible to eliminate one more costate variable at the initial time.

As in Ref. 11, two parameters ξ and ω are chosen to determine the values of the costate function at the initial time. They are chosen as follows:

$$\xi = H_1(t_0)$$

$$\sin\omega = \lambda_\theta(t_0), \qquad \cos\omega = v(t_0)\lambda_v(t_0)$$

where $\omega \in (-\pi, \pi)$ and angle $\omega = \gamma(t_0)$ if $|\omega| < \gamma_0$.

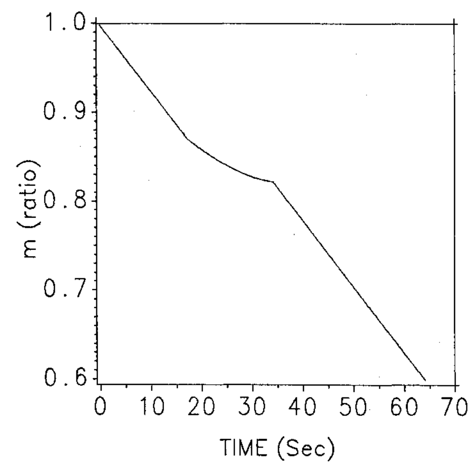

Fig. 5 Variation of mass with time.

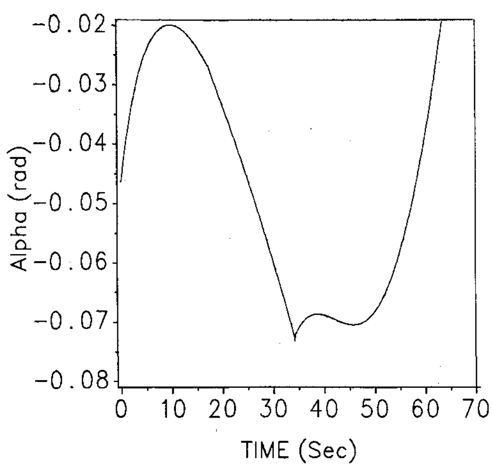

Fig. 6 Variation of angle of attack with time.

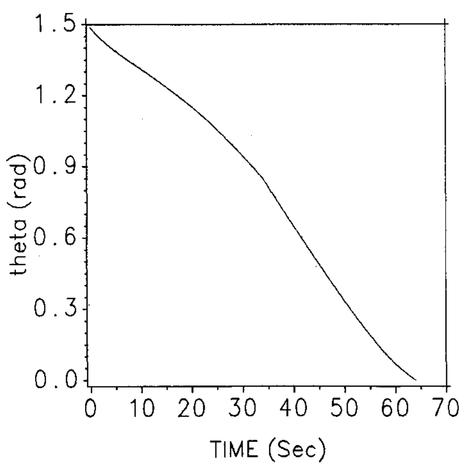

Fig. 4 Variation of flight-path angle with time.

Composite Optimal Trajectory

In conjunction with the solution of the boundary-value problem, good-guess values of the parameters ξ and ω were obtained from Ref. 11. The numerical computation is as follows: from the good-guess values of ξ and ω, the initial values of the costate manifold are established. The state-Euler system is integrated forward with $P=P_{max}$ until the switching function vanishes. The existence of a singular subarc as a section of optimal trajectory, without which the Newton's method of solution of the boundary-value problem does not converge, is established in Ref. 11.

The type of parameterization described does not yield both $H_1=0$ and $\dot{H}_1=0$ identically. This is due to the additional constraint imposed by choosing $\lambda_\theta(t_0)=\sin\omega$ and $\lambda_v(t_0)=\cos\omega/v(t_0)$. Hence, it is not possible to get $\dot{H}_1=0$ simultaneously when the switching function becomes zero, by small variations of the parameters ξ and ω. However, for a given ω, one may choose ξ so that when the switching function becomes zero, its first time derivative is closest to zero.

At the end of the integration interval at which H_1 becomes zero, the costate variables must be such as to produce both H_1 and $\dot{H}_1$ equal to zero, maintaining the Hamiltonian at zero. This may be accomplished by simultaneously solving the equations

$$H_0=\lambda_r \sin\theta+\lambda_\varphi r^{-1}v\cos\theta-\lambda_v(m^{-1}D+r^{-2}\sin\theta)$$

$$+\lambda_\theta v^{-1}m^{-1}L+\lambda_\theta\cos\theta(r^{-1}v-r^{-2}v^{-1})=0$$

$$H_1=\lambda_v m^{-1}\cos\gamma+\lambda_\theta v^{-1}m^{-1}\sin\gamma-\lambda_m c=0$$

$$\dot{H}_1=G_1\cos\gamma+G_2v^{-1}\sin\gamma-G_3c^{-1}=0$$

with λ_v, λ_θ, and λ_m as the variables and the rest of the state and costate with the values obtained at the end of integration with $P=P_{max}$.

The state-Euler system of equations is integrated backwards, using the corrected values of λ_v, λ_θ, and λ_m, to the vicinity of the initial state of the system. The error of the state variables thereby obtained was found to be within the range of tolerance specified for integration.

A singular trajectory along which H_1,$\dot{H}_1$ vanish is plotted from the moment at which H_1 goes to zero. The singular thrust is obtained from Eq. (17). The integration is continued along the singular arc, checking that the Kelley condition and the Goh condition are satisfied. Moreover, the inequality constraint $0\leq P_0\leq P_{max}$ must not be violated.

The exit from the singular arc is made at a suitable time such that a maximum-thrust subarc yields the specified final conditions. This is not trivial, since three states are specified at final time. The mass varies linearly while the flight-path angle variation is nonlinear with respect to time. The exit from the singular arc can be adjusted so that at terminal time the mass constraint and the flight-path angle constraint are satisfied. The radial coordinate at final time is checked. Violation of this constraint demands a change in entry time to the singular arc, implying a variation in the ω chosen. Once the final conditions on the state variables have been satisfied, the final value of the costate variable λ_v is used to scale the values of all the components of the costate manifold throughout the trajectory. This ensures the satisfaction of the transversality condition $\lambda_v(t_f)=1$.

On attaining the state-Euler solution, the values of the Kelley condition parameter K along the singular arc are examined. These were found to be positive all along the subarc, ensuring satisfaction of the Kelley condition. The matrix R formed in the Goh condition was found to be negative semidefinite along the singular subarc, signifying fulfillment of the Goh criterion. Details are in Ref. 20. Thus, the necessary conditions for optimality are met along the singular subarc. Moreover, the classical Clebsch condition that matrix H_{uu} be negative semidefinite is satisfied along the singular subarc.

By variation of the parameters ξ and ω, one can obtain different final conditions. In cases of singular subarcs present in the optimal solution composite, variation of final states can be obtained by changing parameters determining the moment of entry into and departure from the singular subarc.

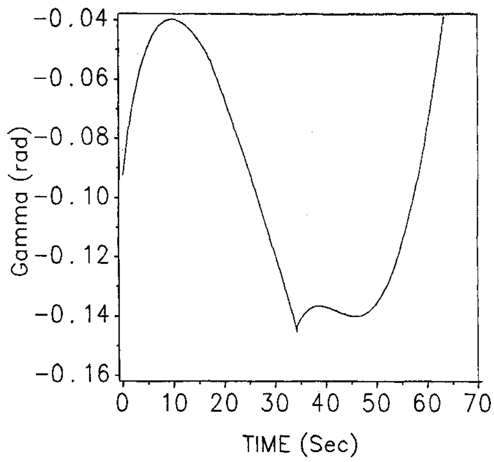

Fig. 7 Variation of thrust-vector angle with time.

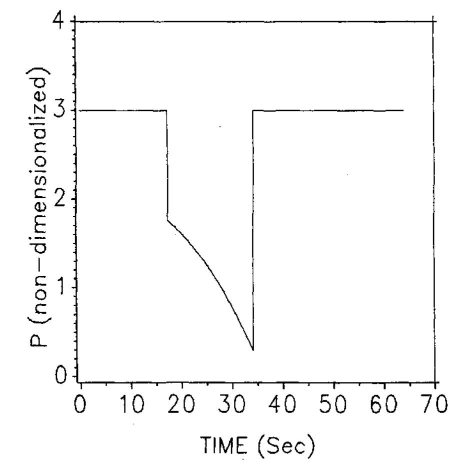

Fig. 8 Variation of magnitude of thrust with time.

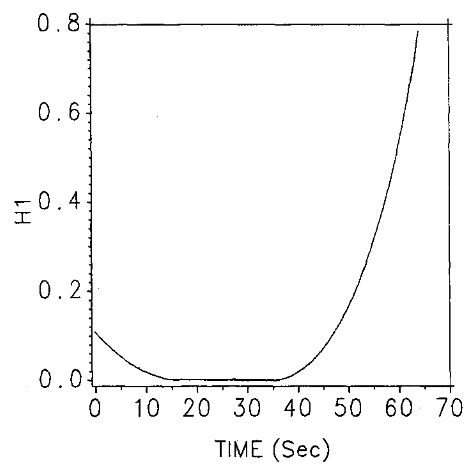

Fig. 9 Variation of switching function with time.

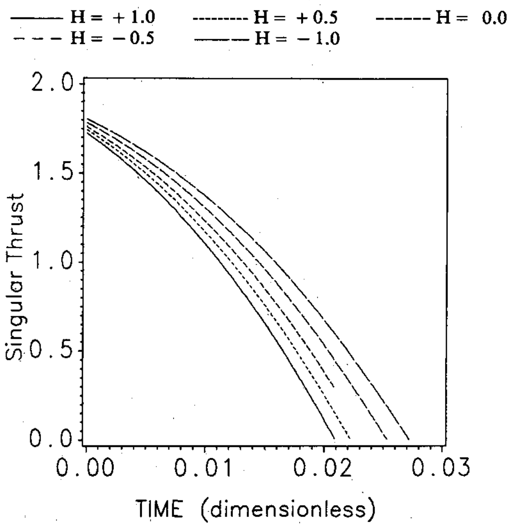

Fig. 10 Variation of singular thrust with time for various H.

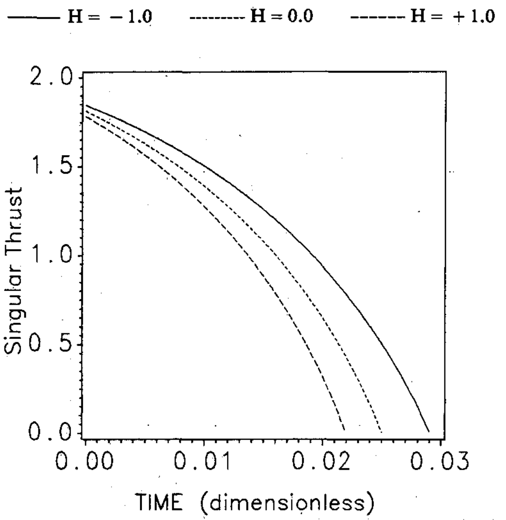

Fig. 11 Variation of singular thrust with time for various H (thrust-along-the-path modeling).

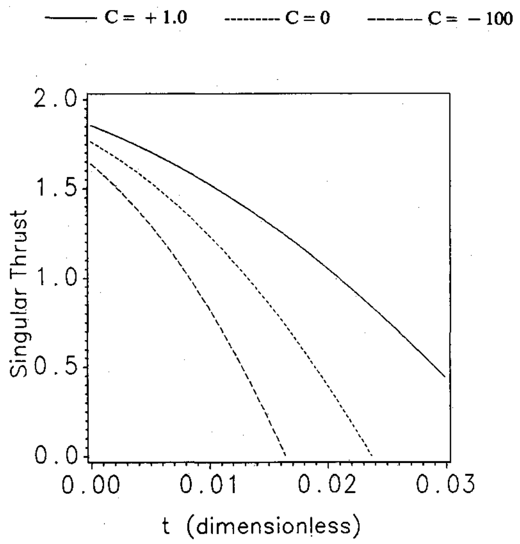

Fig. 12 Singular thrust with time for range-angle specified problems.

Figures 1–9 show the variation of state, control variables, and switching function with time, from initial to final time. The solution obtained is, within roundoff, that of Ref. 11.

Singular Arc with Terminal Time and Range Constrained

A family of singular subarcs emanating from the state at entry into the singular subarc for the terminal-time-open problem was found for $H>0$ (minimum-time problem) and $H<0$ (maximum-time problem). Any positive value of H can be scaled to unity due to the homogeneity of the costate system, and any negative value of H can be similarly scaled to negative unity. The Kelley condition, Goh condition, and Clebsch condition were checked along the subarc and found to be satisfied. Figure 10 exhibits the variation of singular control thrust with time for $H=-1.0$, -0.5, 0.0, $+0.5$, $+1.0$.

Another family of singular arcs was generated for thrust along the path, i.e., control $\gamma=0$. The minimum-time, maximum-time, and time-free members were similarly examined, and optimality was confirmed. Figure 11 shows the variation of singular thrust with time for $H=+1$, $H=-1$, and $H=0$.

The last case studied had the costate $\lambda_\varphi=C$ for various values of the constant C. This is equivalent to a problem with final range angle specified. The necessary conditions were derived and optimality ensured. Figure 12 displays the variation of singular thrust with time for $\lambda_\varphi=C$, $C=0$, 100, -100.

Conclusion

The variable-thrust rocket trajectory of Ref. 11 was examined and the Kelley condition, Goh condition, and classical Clebsch condition were checked. The necessary conditions were satisfied, insuring optimality over short lengths of arc.

Representative singular subarcs were generated for the minimum-time version of the problem, the maximum-time problem, and problems with final range angle specification. The necessary higher-order conditions were found to be satisfied. Similar results were obtained for a thrust-along-the-path model.

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